LYCOMING COLLEGE

WILLIAMSPORT, PENNSYLVANIA

DEPARTMENT OF MATHEMATICAL SCIENCES

Fractals and Convergence of Sequences of Sets

by

Lindsey M. Carr

An honors project thesis submitted to the Honors Committee in partial fulfilment of the requirements for graduation with departmental honors in Mathematics from Lycoming College

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Acknowledgements

It is funny how the beginning is normally the last written. I can now look back at this project and feel a sense of relief because I made my oh-so-close deadline, a sense of accomplishment because I was able to complete a project that I can be proud of, and a sense of sadness because I have grown attached to this project and I will miss the research, the learning, and the long hours of frustrating Dr. de Silva.

First and foremost, I must give my sincerest thanks to Dr. de Silva for his patience, time and energy, and flexibility working around my schedule. I have never learned so much in such a small period of time. If only my memory could retain all the facts! I have found working on this project has strengthened my foundations in Mathematics and reinforced those important concepts from Real Analysis. Again, I cannot thank you enough for all the hard work that you, too, have put forth.

I could not have done this project without my Mom and Dad. Thank you for instilling determination and hard work upon me. Thank you for all the pep talks, Mom, that helped me through all the obstacles!

Furthermore, I would like to give my thanks to the Department of Agriculture that gave me the means to be able to take this class. I would especially like to thank Sherri and Peggy for their fun-loving attitude despite the funny faces they made at my pictures of Heighway's Dragon and for encouraging me to work hard and do well.

Thank you- to all those who encouraged me through my project and listened to the updates. Thank you, Tony for listening to mini-lessons on fractals and for proofreading my paper, Carolyn for peer reviewing, and Rob for debugging advice. Thank you to the bubbly ladies of B. Moss and Northcentral Pennsylvania Conservancy.

Finally, thank you to my committee of four wonderful professors. I could not have asked for a better selection of professors. Thank you for your help and advice, and

your cooperative schedule to meet during the summer. Thanks for popping your head in and giving pieces of advice. It was always greatly appreciated.

Like I said before, I am writing my preface last. I don't think I fully understood the beginning until I reached the end. Working on this project reinforced all those years of Math classes that I have taken. This project has its roots in Differential Equations as well as Real Analysis. Because of this, I find myself a lucky Math major. Working on this research ensured that I fully understood the concepts and techniques studied in those classes. For this, I must thank my professors that challenged me in these courses as well as all my Math and Computer Science courses.

I have been inspired by my professors at Lycoming College, and for this reason I decided to embark on an honors project. I hope that this inspiration and enthusiasm is reflected in my paper.

CONTENTS

Abstract:

This honors project compares the properties of several alternative types of convergence, namely Uniform Convergence, Hausdorff Distance, and Inner and Outer Limits, and their application to fractals.

1 Fractals

A person sits on the couch watching television. On the television screen appears the same person sitting on a couch watching television. This spiral of images continues on and demonstrates the basic idea of self-similarity, which is a central property of a fractal [Mandelbrot]. A fractal is a fragmented geometric shape that can be divided into parts, each of which is a smaller replica of the whole.

1.1 Iterated Function Systems

Iterated Function Systems (IFS) are a way of constructing fractals. This is easy to illustrate with the following example. Consider a triangle as our first set. We can apply a process of reducing the sides of the triangle to half size and make three copies which we connect to form another triangle whose corners are at the midpoints of the sides of the original triangle. This triangle becomes the second in the sequence of sets. We can apply this process repeatedly to create a sequence of sets, which approaches Sierpinski's Triangle as seen in the figure 1.1 below.

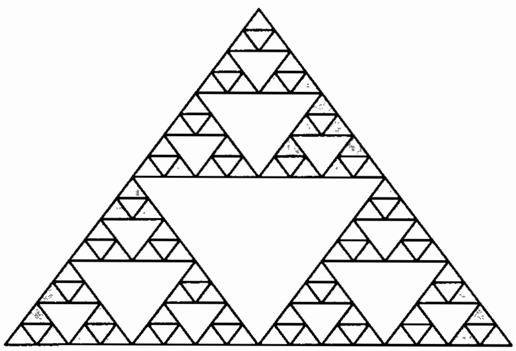


Figure 1.1: The fifth iteration in Sierpinski's Triangle

In general, we start with an arbitrary set and a collection of functions, each of which is a combination of reflection, translation, rotation, and/or dilation. Combining this collection of functions into a single compound function, F, we apply F to the original set to generate a new set. Such a compound function, F is known as an Iterated Function Systems when applied to sets in order to generate fractals [Schienerman].

1.2 An Example of an IFS: Heighway's Dragon

An interesting example of Iterated Function Systems is that of Heighway's Dragon. Initially, we start with two congruent line segments forming a right angle as shown in Figure 1.2 below. Let's call the leftmost point of the initial figure, *the tail*, the lowest point, *the foot*, and the rightmost point, *the head*. To this set we apply two functions, F_1 and F_2, which comprise the Iterated Function System.

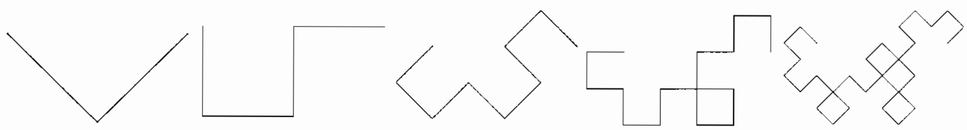


Figure 1.2: Sequence of sets that approaches Heighway's Dragon

F_1 consists of a 45° rotation clockwise and dilation by a factor of $\sqrt{½}$ in such a way that the tail of the modified figure is placed where the tail of the original was, and the head of the reduced figure is placed where the foot of the original used to be.

F_2 consists of a 135° rotation clockwise and dilation by the same factor of $\sqrt{½}$ in such a way that the head of the new modified piece is touching the head of the piece created by F_1, which now forms the foot of the new dragon. The tail of the piece created by F_2 forms the head of the next in the sequence.

Taking the union of F_1 and F_2 forms F. More specifically, the result of F applied to the set S is $F(S) = F_1(S) \cup F_2(S)$.

This sequence of figures above (Figure 1.2) shows the initial set and the results of four applications of the Iterated Function System F. Heighway's Dragon is what is obtained when this process is continued indefinitely [Edgar].

The fractals that we have been discussing are called deterministic fractals. Deterministic fractals are those that are generated according to some specific rule, such as an Iterated Function Systems. Fractals that are randomly generated are refered to as nondeterministic fractals.

Figure 1.3 shows a rather elaborate depiction of the tenth set in the sequence for Heighway's Dragon with sets one to nine shown behind it.

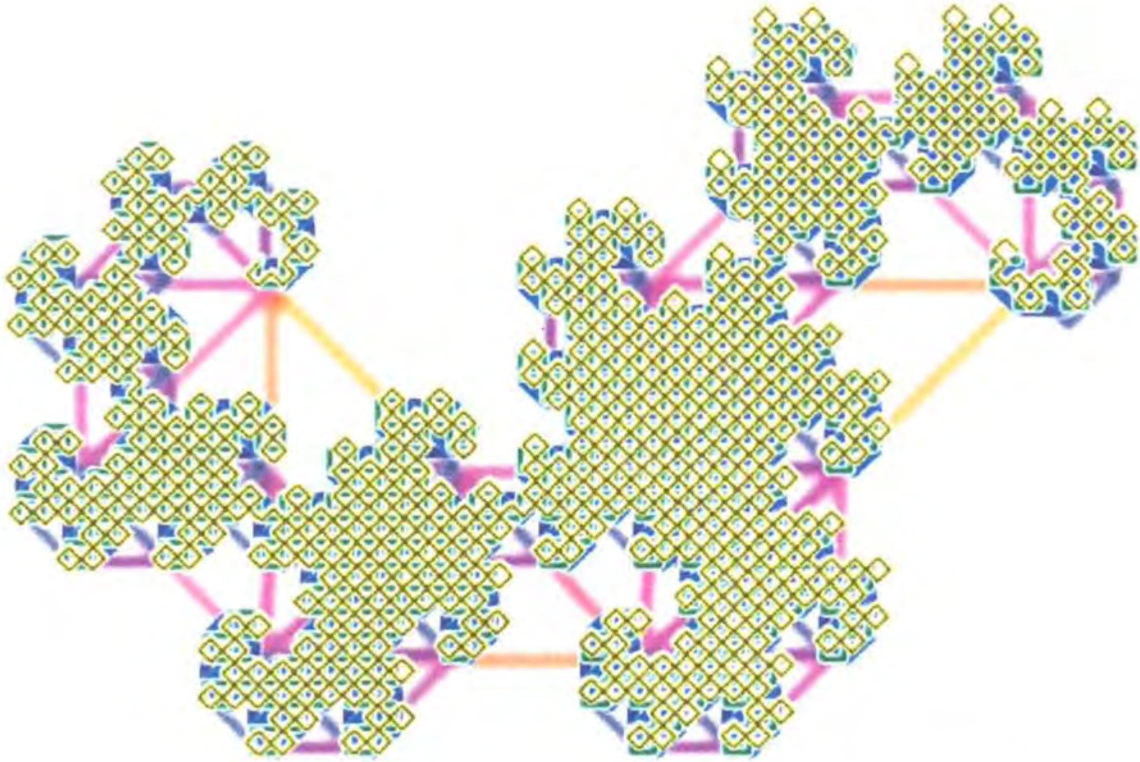


Figure 1.3: Tenth figure in the sequence for Heighway's Dragon (the previous nine iterations are shown beneath it).

2 Fractals as Limits of Sequences of Sets, Convergence

As with Sierpinski's Triangle, Heighway's Dragon is the limit of a certain sequence of sets. This project focuses on the phenomenon of convergence of sets.

In many examples of fractals obtained as limits of sequences, the existence of a limit is obvious. For example, the so-called Koch snowflake (shown in Figure 2.1) is simply defined by taking the union of the sequence that is used to define it.

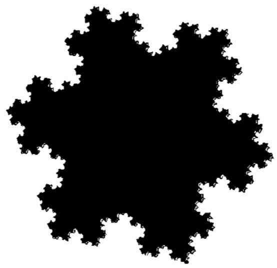


Figure 2.1: Koch Snowflake --generated using Geometer's Sketchpad.

Also, Sierpinski's Triangle is made up of the intersection of the sequence that defines it. However, in the case of Heighway's Dragon, we modify each set in the sequence to find its successor, and the two sets have very few points in common. Therefore, the idea of convergence of sets is an important one to study in order to understand the definition of a fractal as defined by an Iterated Function System from a technical point of view.

Before examining the convergence of sets, we will begin with an overview of convergence of sequences of simpler objects such as sequences of points and functions.

3 Metric Topology of Sets

The natural setting to describe the convergence of a sequence of points is in a *metric space.* We will describe below the essential features of a metric space, which is effectively just a *distance function*. Using this distance function, we define the convergence of a sequence of points. It is then possible to define a useful kind of convergence of functions, namely *uniform convergence.* Finally, we look at two alternative methods for analyzing the convergence of a sequence of sets of points.

Various kinds of convergence have different applications. The type of convergence used reveals something of the properties of the limit and their relationship to the properties of the sets of the sequence, for instance whether the limit is closed or bounded.

3.1 Metric spaces

Suppose we have a set X, and a function $d : X \times X \to \mathbb{R}$ that satisfies the following properties:

1. $d(x,y) \geq 0$, for every x and y in X, and $d(x,y)=0$ if and only if $x=y$ [positivity],
2. $d(x,y) = d(y,x)$, for every x and y in X [symmetry], and finally,
3. $d(x,z) \leq d(x,y) + d(y,z)$ [Triangle inequality].

Such a function d is called a *metric* for X, and the system (X, d) is called a *metric space.*

3.2 Balls and Boundedness

Balls are objects in metric spaces. Let (X,*d*) be a metric space. Let *p* be a point in X, and let *r* be any positive number. Then we define the ball of *radius r* around point *p* to be the set consisting of all points in X whose distance from *p* is less than *r*.

Some important concepts that we can define immediately are:

Bounded Sets: A set *S* is said to be *bounded* if and only if it is a subset of some open ball.
Open Sets: A set *S* is said to be *open* if and only if it is a union of open balls.
Closed Sets: A set *S* is said to be *closed* if and only if its complement is an open set.
Compact Sets: For the purposes of this paper, in which the setting is $\mathbb{R}^n$, there is a simple definition of compactness: A set is *compact* if and only if it is closed and bounded.

The following properties of these kinds of sets follow immediately from their definitions.

1. Any union of open sets is also open. A finite intersection of open sets is open. Any intersection of closed sets is closed, and finite unions of closed sets are closed.
2. Finite sets are bounded and closed, and therefore compact.
3. Finite unions of compact sets are also compact. A closed subset of a compact set is compact [de Silva].

The question then follows: are all fractals compact? The answer is no because not all fractals are closed. Although we can see that Sierpinski's Triangle is closed, others are not. For example, we can define a fractal on the real line from zero to one consisting of the points {0, 1, 1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, 1/16 ... }. This fractal is clearly not closed.

3.3 Convergence (of points): Definitions and properties

Suppose $[u_n]$ is a sequence in some metric space X, and suppose L is a point in X. We say that L is the limit of the sequence $[u_n]$ if and only if every open ball B around L contains all the points of $[u_n]$ except a finite number.

The limit of a bounded sequence lies within the same bounds as the sequence itself. Furthermore, if the points of a sequence lie in a closed set F, the limit also lies in F.

3.4 Cauchy Sequence and completeness

Suppose $[u_n]$ is a sequence which has no obvious limit. In this case, we cannot apply the definition of convergence, and a different approach must be taken. The Cauchy Criterion is used in such cases. The Cauchy Criterion examines whether a sequence has the potential to have a limit. For every metric space, if a sequence is not Cauchy it will definitely not converge. In a few special spaces, we can infer more.

A sequence $[u_n]$ is said to satisfy the Cauchy Criterion if for every positive ε, there exists an integer N such that for every pair of integers p and q greater than N, we have $d(u_p, u_q) < \varepsilon$. [Such a sequence is said to be a Cauchy sequence, or simply Cauchy.]

Theorem 1: If a sequence is convergent, then it is Cauchy.

The converse is not always true. A metric space in which every Cauchy sequence is also convergent is said to be a complete metric space. A few well-known complete metric spaces include the real numbers with the distance function $d(x, y) = |y - x|$ given by the absolute value, and more generally Euclidean n-space with the Euclidean distance. It is also interesting that $K(S)$ defined below is complete.

4 The Hausdorff Distance

Hausdorff Distance is a way to describe convergence of sets. The Hausdorff Distance is the distance between two compact sets.

4.1 Definitions and Illustrations

Suppose (S, d) is a metric space. Let $K(S)$ denote the collection of all compact subsets of S. For any J in $K(S)$, define $N_R(J) = \bigcup\{B_R(p) \mid p \in J\}$, using the metric in S. We call this set the *R-expansion of J.*

Next we define the distance-function D as follows for any pair of sets U, V [Edgar]:

$$D(U,V) = \inf\{R>0 \mid V \subseteq N_R(U), \text{ and } U \subseteq N_R(V)\}.$$

It is important to remember that we are dealing with two metric spaces. The first is the metric space (S, d). The second is $(K(S), D)$.

In Figure 4.1, an elephant and a giraffe demonstrate the basic concept of the Hausdorff Distance. Each represent compact sets. For the sake of illustration, we have placed an R-expansion of a certain radius around the giraffe, depicted in yellow. Similarly, we placed an R-expansion of a certain radius around the elephant, shown in blue.

Figure 4.1: The Hausdorff Distance between two bounded sets, consisting of an elephant and a giraffe

From the diagram, we see that the region around the giraffe does not sufficiently cover the elephant, nor does the blue area cover the giraffe entirely. Thus, we must make the radius larger until both sets are adequately covered. The Hausdorff Distance is the least such radius.

Theorem 2 [Scheinerman]: *Let A, B, C, D be sets in K(S). Then*

$$D(A \cup B, C \cup D) \leq \max \{D(A, C), D(B, D)\}.$$

4.2 Is K(S) complete?

We want to prove that the metric space $K(S)$ is complete. This fact will then help in our proof of convergence in cases where the limit is not obvious.

Theorem 3: Suppose (S, ρ) is a complete metric space. Then the space $(K(S), D)$ is also complete.

[$K(S)$ is the set of all compact subsets of S and is associated with the Hausdorff Distance.]

Proof [following Edgar]: Suppose $[A_n]$ is a Cauchy sequence in $K(S)$. We must show that $[A_n]$ converges. Let $A = \{x \mid \text{there is a sequence } [x_k] \text{ with } x_k \in A_k \text{ and } x_k \to x\}$. A consists of the limits of $[x_k]$. Now, we must show that A_n converges to A.

Let $\varepsilon > 0$ be given. Then, there exists a natural number N such that $p, q \geq N$ implies $D(A_p, A_q) < \varepsilon/2$. This is the Cauchy Criterion applied to $[A_n]$ in the hypothesis.
Let $p \geq N$. We must show that $D(A_p, A) \leq \varepsilon$.

If $x \in A$, then we know that there is a sequence $[x_k]$ with $x_k \in A_k$ and $x_k \to x$. Thus, for large enough q, we have $\rho(x_q, x) < \varepsilon/2$. This is from the definition of convergence. Also, if $q \geq N$, then (since $D(A_q, A_p) < \varepsilon/2$) there is a $y \in A_p$ with $\rho(x_q, y) < \varepsilon/2$, thus
$\rho(y, x) \leq \rho(y, x_q) + \rho(x_q, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This says that there exists some y in A_p within ε distance of x. Therefore, $A \subseteq N_\varepsilon(A_p)$. [$N_\varepsilon(A_p)$ denotes the union of all ε-balls around points of A_p.] We have shown that ε-balls around points in A_p cover A.

Now suppose $y \in A_p$. Choose integers $k_1 < k_2 < \ldots$ such that

$k_1 = p$ and $D(A_{k_j}, A_m) < \dfrac{\varepsilon}{2^j}$, for all $m \geq k_j$ because $[A_k]$

is Cauchy. For example, choose k_1 such that $k_1 = p$ and

$D(A_{k_1}, A_m) < \frac{\varepsilon}{2}$ for all $m > k_1$. Next, choose k_2 such that

$D(A_{k_2}, A_m) < \frac{\varepsilon}{4}$ for all $m > k_2$. In general, choose k_j such

that $D(A_{k_j}, A_m) < \frac{\varepsilon}{2^j}$, for all $m \geq k_j$.

Then define a sequence $[y_k]$ with $y_k \in A_k$ as follows: define $y_1 \ldots y_{p-1}$ arbitrarily in $A_1 \ldots A_{p-1}$, and define $y_p = y$.

For $k_j < k \leq k_{j+1}$, choose $y_k \in A_k$ such that $\rho(y_{k_j}, y_k) < \frac{\varepsilon}{2^j}$.

Then y_k is a Cauchy sequence, so it converges. Let x be its limit. Thus, $x \in A$. We have $\rho(y, x) = \lim \rho(y, y_k) < \varepsilon$. Thus $y \in N_\varepsilon(A)$. This shows that $A_n \subseteq N_\varepsilon(A)$.

So we have $D(A, A_n) \leq \varepsilon$, Therefore $[A_n]$ converges to A.

It remains to show that A is in fact compact. We omit the proof of this fact. The proof is to be found in *Measure, Topology, and Fractal Geometry* by Edgar. ■

5 Convergence from the point of view of Fractals

The type of convergence that may be applied to fractals depends on the type of fractal. For example, if we look at a fractal such as Sierpinski's Triangle, we are looking at a sequence of sets, and therefore are forced to use Hausdorff Distance, or Inner and Outer Limits, which will be described below. In the case of a fractal that is defined using a sequence of curves we can also use the approach of uniform convergence. Below we examine these types of convergence.

5.1 Convergence of Functions

There are two concepts of convergence of functions, namely *pointwise* convergence and *uniform* convergence. Because pointwise convergence of a sequence of functions does not tell us anything about the properties of the limit-function, we focus on uniform convergence.

Definition: Uniform Convergence

Consider a set S in a metric space X, and a metric space Y with metric d. Suppose we have a sequence $[f_n]$ of functions from S into Y, i.e, $f_n : S \to Y$. Let g be another function from S into Y, i.e, $g: S \to Y$. We say that $[f_n]$ approaches g uniformly on the set S if and only if for every $\varepsilon > 0$, there exists a positive integer N such that for every $k > N$, and for every x in the set S, $d(g(x), f_k(x)) < \varepsilon$. An important theorem associated with uniform convergence is [Edgar]:

Theorem 4: Let $[f_n]$ be a uniformly Cauchy sequence f_n: $S \to Y$, relative to the metric space (Y,d). Then $[f_n]$ is convergent provided (Y,d) is complete.

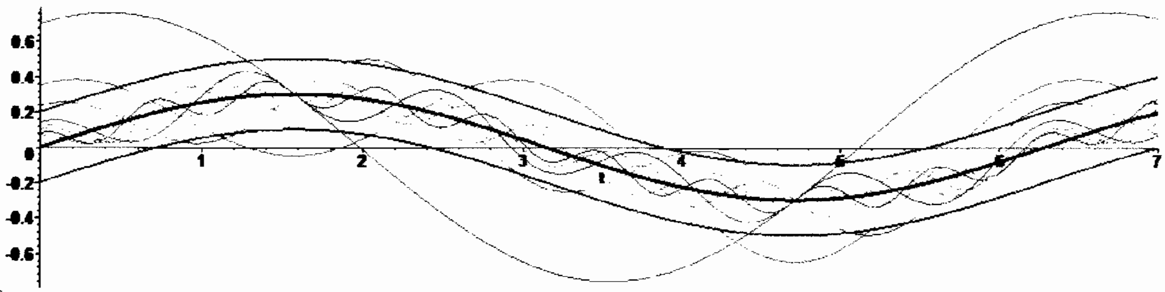


Figure 5.1: Demonstrates uniform convergence

Visually speaking, if we place a tube of arbitrary radius ε around the graph of the limit-function g, it must be the case that the tube contains the graphs of all but a finite number of the functions f_n [see above]. Figure 5.2 is an example that displays pointwise convergence without uniform convergence. The limit function is the x-axis between -1 and 1, as well as the two points (1, 1) and (-1, -1). Clearly, this is not a continuous function although each function in the sequence is.

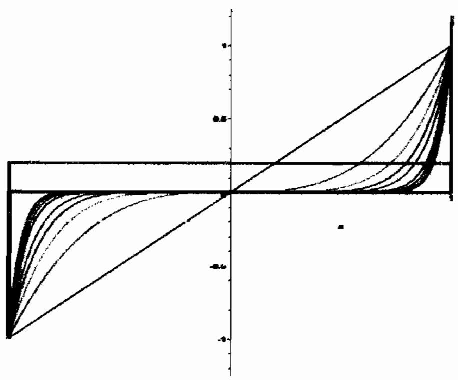


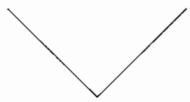
Figure 5.2: Demonstrates non-uniform convergence, more specifically pointwise convergence

Theorem 5: If the functions f_n are all continuous on S, then their uniform limit g will also be continuous on S.

5.2 *Uniform Convergence of Heighway's Dragon*

We examine Heighway's Dragon as a sequence of functions rather than a sequence of curves. Let us call this sequence Heighway's Sequence. Each dragon in the sequence can be written using parametric equations. For example, the first curve pictured in Figure 5.3 is:

Figure 5.3: First set in the sequence of Heighway's Dragon



$$x(t) = t; \quad y(t) = \begin{cases} -t, & 0 \le t \le \frac{1}{2} \\ t-1, & \frac{1}{2} < t \le 1 \end{cases}$$

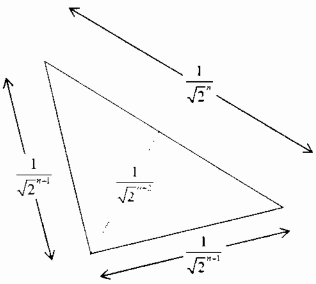
The others follow in the same manner.

In order to prove Heighway's Sequence is uniformly convergent, we need to show it is uniformly Cauchy. Following the definition for uniform convergence, we must show that for every $\varepsilon > 0$, there is a positive integer N such that for every $n, m > N$, and for every t in the set $0 \le t \le 1$, $d(F_n(t), F_m(t)) < \varepsilon$.

Next we show that $d(F_n(t), F_{n+1}(t)) < \frac{1}{\sqrt{2}^{n+2}}$, from which we can show the inequality in the previous paragraph.

When we defined Heighway's Dragon, we said that both functions in its associated Iterated Function System contained a dilation of $\sqrt{1/2}$. This means that if looking at the n^{th} set in the sequence, the length of each line segment would be $\frac{1}{\sqrt{2}^n}$. In the same manner, the length of the $n+1$ set would be $\frac{1}{\sqrt{2}^{n+1}}$. Thus, finding the distance between the n and the $n+1$ set would be the altitude, or $\frac{1}{\sqrt{2}^{n+2}}$.

Figure 5.4: A typical segment of Dragons n and $n+1$.



Assume $m > n$. Then, by applying the triangle inequality, $d(F_n(t), F_m(t)) \leq d(F_n(t), F_{n+1}(t)) + d(F_n(t), F_{n+2}(t)) + ... + d(F_{m-1}(t), F_m(t))$. Substituting, we see it equals $\frac{1}{\sqrt{2}^{n+2}} + \frac{1}{\sqrt{2}^{n+3}} + ... + \frac{1}{\sqrt{2}^{m+1}}$, which can be shown, using geometric series, to be less than $\frac{2}{\sqrt{2}^{n}}$.

From these facts, we can proceed with the proof to show that Heighway's Sequence is uniformly Cauchy. Let $\varepsilon > 0$ be given. Let N be chosen such that $\frac{2}{\sqrt{2}^{N}} < \varepsilon$.

Then, for every $m, n > N$, $\max\{d(F_m(t), F_n(t)) \mid 0 \leq t \leq 1\} < \frac{2}{\sqrt{2}^{\max\{n,m\}}} < \frac{2}{\sqrt{2}^{N}} < \varepsilon$.

Therefore, Heighway's Sequence is uniformly Cauchy.

Note that Theorem 4 suggests since $\mathbb{R}^2$ is complete, if a sequence of functions is uniformly Cauchy, it is uniformly convergent. Therefore, the limit function is continuous.

Thus, Heighway's Sequence is uniformly convergent, and therefore Heighway's Dragon exists.

6 Convergence of Sets

With deterministic fractals, applying the processes described above depends on the existence of a limit of a sequence of sets, or curves. There are several methods of defining such limits. However, studying a sequence of sets using a particular limit-definition does not guarantee that the limit exists.

One of the most useful methods of defining limits of sets uses the setting of a metric space, and the so-called Hausdorff metric defined previously.

6.1 Hausdorff Distance-review

Recall that the Hausdorff distance only applies to compact sets. Suppose U, V are both compact. Then $D(U,V)$ = the smallest radius r such that U is a subset of the r-expansion of V and vice-versa.

To apply this metric to the problem of convergence of sets is straightforward. Suppose we consider $\mathbb{R}^2$. First, let $K(\mathbb{R}^2)$ be the set of all compact sets in $\mathbb{R}^2$. The Hausdorff distance is a natural metric for this space. Convergence is defined as follows: Let $[S_n]$ be a sequence of sets and let L be another set. We say that L is the limit of the sequence $[S_n]$ if and only if every open ball B around L contains all the sets $[S_n]$ except a finite number.

Consider Figure 6.1 below.

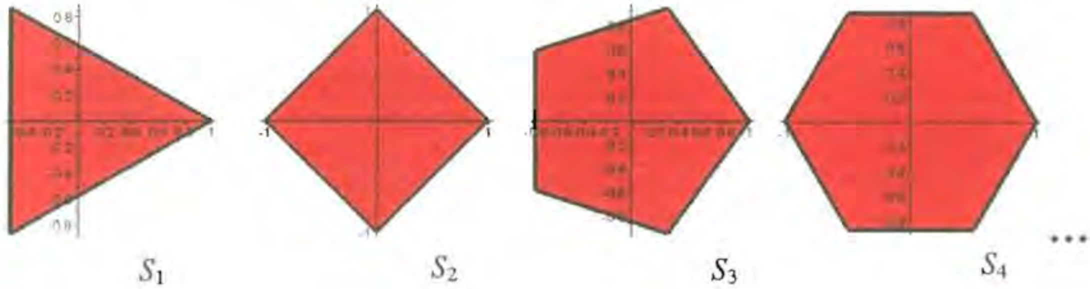


Figure 6.1: Sequence of sets in $\mathbb{R}^2$ that appear to converge to a circle

The larger the number of sides of the polygons, the more closely they approach a circle. Let us check whether this sequence of figures tends to the limit which intuitively appears to be a circle using Hausdorff distance.

The Hausdorff distance $D(S_1,C)$ between the triangle and the circle C is ½. To see this, we fill the triangle with open balls of radius ½, and observe that it covers the circle. Similarly, we cover the circle with open balls of radius ½, and observe that it covers the (original) triangle. It is also clear that circles with at least a radius of ½ are required.

In a similar manner, we find $D(S_2,C)$ which is $1-\sqrt{½}$, is approximately 0.2929. In general, the Hausdorff distance from the n^{th} set to the circle is:
$D(S_n, C)=1-$ (the closest point of the boundary of the polygon to the origin)$=1-\cos(\pi/n)$. As n increases, the expression above approaches 0, and thus we conclude that the limit of the sequence relative to the Hausdorff metric is indeed the unit circle.

Let's apply the Hausdorff distance approach to Heighway's Dragon (Figure 6.2). We consider the dragon whose tail is positioned at (0, 0) and head at (1, 0).

Figure 6.2: Heighway's Dragon

It was the case in the previous example that we were able to find a particular limit, namely the unit circle. While the limit was obvious in that example, we are not able to predict the exact limit-set in the case of Heighway's Dragon. Since we are not able to guess a limit, we must resort to using the *Cauchy Criterion*. [If a sequence is convergent, then the Cauchy Criterion is satisfied. However, the converse is not necessarily true.] Recall the Cauchy Criterion for a sequence $[D_n]$ which states that for every positive ε there must correspond an integer N such that for all $p, q > N$, $D(D_p, D_q) < \varepsilon$.

In the present example, we can show that for all $n > m$, $D(D_n, D_m) = 1/(\sqrt{2})^{m+1}$.

Figure 6.3A: Balls around first set cover the second set

Figure 6.3B: Balls around second set cover the first set

The figures above explain this distance calculation for the case of D_1 and D_2. The figure on the left depicts a ball of radius $2^{-1.5}$ sliding along D_1, just barely covering D_2. The figure on the right depicts a ball of the same radius sliding along D_2, comfortably covering D_1. Furthermore it is clear that expansions of any radius > $2^{-1.5}$ of each set would certainly cover the other, while any expansion of radius < $2^{-1.5}$ about D_1 will fail to cover all of D_2. In the general case, the radius is of great importance and determined by the Hausdorff definition.

It is easy to see that the distances of D_n to all succeeding dragons are the same, namely $2^{-½(n+2)}$. The proof is by induction on *n*.

It is easy to see that $D(D_1, D_{1+r}) = \frac{1}{2\sqrt{2}}$.

Suppose $D(D_k, D_{k+r}) \leq \frac{1}{2^{\frac{k+2}{2}}}$. [We must show that $D(D_{k+1}, D_{k+1+r}) \leq \frac{1}{2^{\frac{k+1+2}{2}}}$.]

Suppose D_{k+1} = a figure of two arbitrary parts, say *A* and *B*, and D_{k+r+1} = a figure of two arbitrary parts, say *C* and *D*. Theorem 2 states that, for all compact sets *A, B, C, D*, $D(A \cup B, C \cup D) \leq \max\{D(A, C), D(B, D)\}$. In this particular case, that is $\max\{D(2^{-½} D_k, 2^{-½} D_{k+r}), D(2^{-½} D_k, 2^{-½} D_{k+r})\} = D(2^{-½} D_k, 2^{-½} D_{k+r})$
$= 2^{-½} D(D_k, D_{k+r}) \leq 2^{-½} . 2^{-½(k+2)} = 2^{-½(k+1+2)}$.

Since *k* was arbitrary, it follows by induction, that $D(D_m, D_n) \leq 2^{-½(m+2)}$ for all $n > m$.

Since $D(D_n, D_m)$ can be made less than any positive ε, the sequence of curves that comprise Heighway's Dragon is Cauchy with respect to $K(\mathbb{R}^2)$. Since $K(\mathbb{R}^2)$ Dragon is complete, it follows that the limit exists!

6.2 Inner and Outer Limits

First, let's observe inner and outer limits for a sequence of numbers.

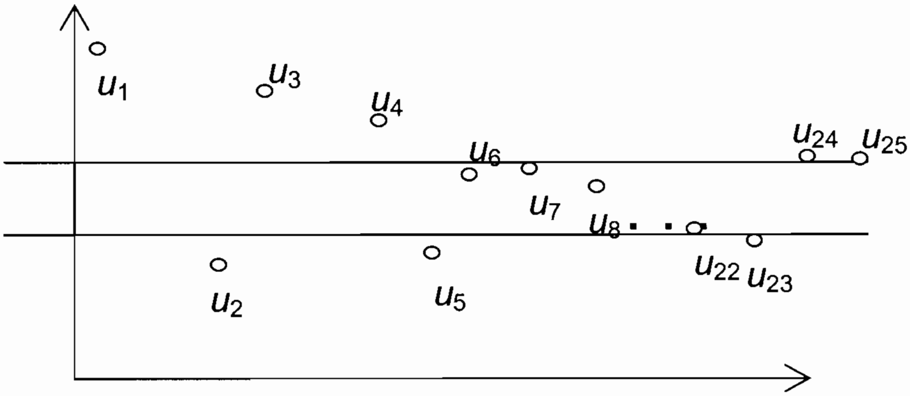


Figure 6.4: Nonconverging sequence in ℝ

This technique of inner and outer limits assumes that sequences are bounded. Consider the sequence illustrated in Figure 6.4. Like this example, the vast majority of bounded sequences do not converge. However, for each bounded sequence, it is possible to define two numbers called its lower and upper limits.

The characteristic property of the lower and upper limits is that they define a certain interval. This interval has the property that every open interval that contains it also contains *all but a finite number of points of the sequence.* Furthermore, our goal is to find the *smallest* such interval.

Let $[U_n]$ be any bounded sequence. First, we must define two associated sequences $[A_n]$ and $[B_n]$ as follows:

A_1 = the infimum of the entire sequence.

A_2 = the infimum of all the terms excluding U_1.

A_3 = the infimum of all the terms excluding U_1 and U_2.

.

.

.

A_n = the infimum of all the terms excluding $U_1 \ldots U_{n-1}$.

Similarly,

B_1 = the supremum of the entire sequence.

B_2 = the supremum of the entire sequence excluding U_1.

B_3 = the supremum of the entire sequence excluding U_1 and U_2.

.

.

.

B_n = the supremum of the entire sequence excluding $U_1 \ldots U_{n-1}$.

By the Order-Completeness Axiom for $\mathbb{R}$, we know that each A_n and B_n exist because the sequence is bounded. Moreover, we know that $[A_n]$ is increasing and bounded and $[B_n]$ is decreasing and bounded. Therefore, we can conclude that both have limits. The limit of $[A_n]$ is called the lower limit, denoted lim inf U_n, and the limit of $[B_n]$ is called the upper limit, denoted lim sup U_n.

For example, looking at our original diagram:

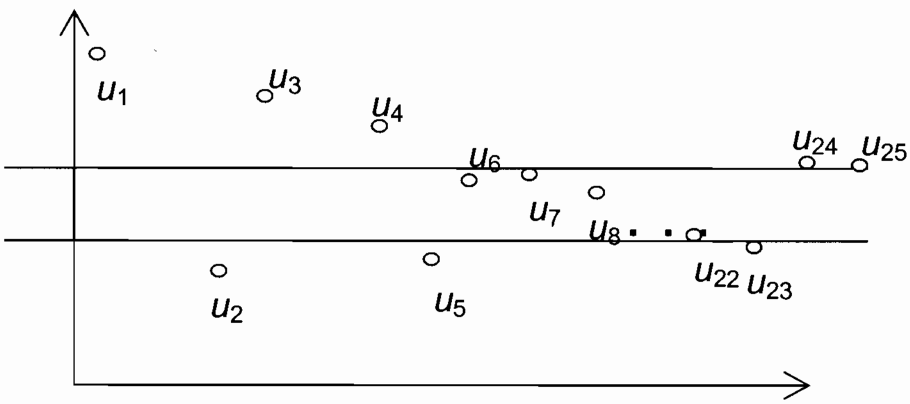


Figure 6.5: Figure 6.4 replicated here for your convenience

$A_1 = A_2$ appear to be U_2

$A_3 = A_4 = A_5$ appear to be U_5

$A_6 = A_7 = \ldots = A_{23}$ appear to be U_{23} .

Similarly, we can see the upper limits as follows:

B_1 appears to be U_1

$B_2 = B_3$ appear to be U_3

B_4 appears to be U_4

$B_5 = B_6 = \ldots = B_{24}$ appear to be U_{24}

B_{25} appears to be U_{25}

From this, we can see that the limit of the As and the limit of the Bs will approach two values that have the properties previously specified.

Taking these ideas a step further, we can extend them to apply to a sequence of sets. To help explain this concept, we will examine a general example.

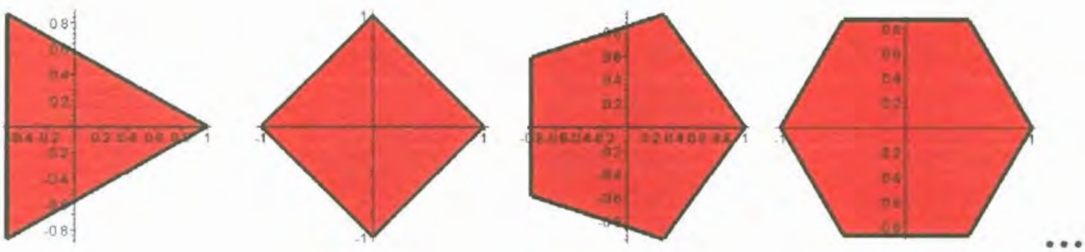


Figure 6.6: Sequence of sets that appear to converge to a circle

As the sequence progresses, the figures possess more sides. The sequence converges to a circle. Instead of using infima and suprema as in lower and upper limits, we explore the notion of inner and outer limits using intersections and unions respectively.

In general, let $[S_n]$ be a sequence of compact sets. First, we must define associated sequences of sets, $[A_n]$ and $[B_n]$ as follows:

A_1 = The intersection of the entire sequence of sets = $S_1 \cap S_2 \cap S_3 \cap S_4 \cap \ldots$

A_2 = The intersection of the entire sequence of sets except S_1 = $S_2 \cap S_3 \cap S_4 \cap \ldots$

.

.

.

A_n = The intersection of all the sets except $S_1 \ldots S_{n-1}$ = $S_n \cap S_{n+1} \cap \ldots$

Analogously, define

B_1 = The Union of the entire sequence of sets = $S_1 \cup S_2 \cup S_3 \cup S_4 \cup \ldots$

B_2 = The Union of the entire sequence of sets except S_1 = $S_2 \cup S_3 \cup S_4 \cup \ldots$

.

.

.

B_n = The Union of all the sets except $S_1 \ldots S_{n-1}$ = $S_n \cup S_{n+1} \cup \ldots$

Observe that the sequence of sets $[B_n]$ is nested (form sets that are gradually smaller), while sequence of sets $[A_n]$ is expanding. Therefore, the limit of the $[B_n]$ is found by taking their intersection, and the limit of the $[A_n]$ is found by taking their union.

In the present case, taking the intersection A_1 of the entire sequence of sets (Figure 6.7A), we see that it is a region near the origin in the form of a pentagon. For B_1, taking the union of the entire sequence of sets (Figure 6.7B), the figure appears to be a solid circle.

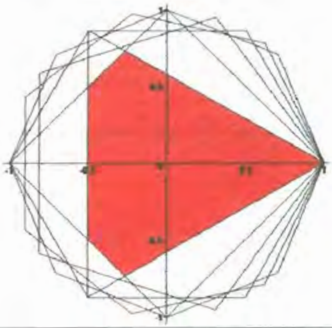


Figure 6.7A: A_1- The intersection of all the sets, generated by *Maple 9*

Figure 6.7B: B_1- The union of all the sets, generated by *Maple 9*

The intersection that corresponds to A_2, however, does not include the triangle S_1, and consequently is larger than A_1, and now appears to be an octagon. On the other hand B_2 remains a circle. [In most examples, the sequence $[B_n]$ would not be constant.] In general,

$$\text{Inner Limit } S_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} S_k\right),$$

and

$$\text{Outer Limit } S_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} S_k\right)$$

If the inner and the outer limit coincide, then we say that the sequence of sets is *convergent,* and the common limit is called The Limit. [Note that inner and outer limits always exists, while the sequence may or may not possess *a limit.*]

In this example, the sequence $[A_n]$ will gradually approach the unit circle, and the sequence $[B_n]$ will also approach the unit circle, hence, the inner- and the outer-limits coincide, and the sequence is convergent.

6.3 Apply to Heighway's Dragon

We can apply this notion of inner and outer limits to Heighway's Dragon.

First, let $[D_n]$ be the sequence of sets that define the dragon. We then define associated sequences of sets, $[A_n]$ and $[B_n]$ as before.

Below in Figure 6.8 we have shown what $B_1 \ldots B_5$ would look like. B_2 is the union of all the dragons except D_1 has been removed. Each figure has one more dragon removed from the previous. As n goes to infinity, the sequence B_n appears to approach Heighway's Dragon.

Figure 6.8: $B_1 \ldots B_5$ for Heighway's Dragon, generated using *Geometer's Sketchpad*

Next we show what the sequence A_n looks like in Figure 6.9. As figures in the sequence are removed, more points are in common with the remaining sets. As we take the union of points in A_n, we end up with a countable union of finite sets, which is also countable.

Figure 6.9: $A_1 \ldots A_4$ for Heighway's Dragon

Therefore, the inner limit and the outer limit appear not to be the same in this case.

Conclusions and possible extensions

Generating a sequence of sets according to some rule produces fractals; the actual fractal is the limit of the sequence of sets. There are a number of methods of determining whether a limit exists, the most useful being that of the Hausdorff Distance. We also explore the use of inner and outer limits. We have discovered that while Heighway's Dragon is a convergent sequence according to the Hausdorff Metric, using Inner and Outer Limits, Heighway's Dragon does not converge. We used Heighway's Dragon as a representative fractal.

As a possible extension, it would be interesting to find whether one type of convergence implies convergence with respect to another. For example, if we knew that a set was inner and outer convergent would it be Hausdorff convergent?

Appendix A

```
/************************************
* Heighway's Dragon                 *
* HeighwaysDragon.java              *
* Author: Lindsey Carr              *
* Date: Summer 2003                 *
* Honor's Project                   *
************************************/

import java.awt.*;
import java.awt.event.*;
import javax.swing.*;

public class HeighwayDragon extends JApplet implements
ActionListener{

  private final int APPLET_WIDTH = 512;
  private final int APPLET_HEIGHT = 512;

  private final int MIN = 0, MAX = 15;

  private JButton increase, decrease;
  private JLabel titleLabel, orderLabel;
  private HeighwayDragonPanel drawing;
  private JPanel appletPanel, tools;

  // set up the components for the applet
  public void init(){
    tools = new JPanel ();
    tools.setLayout(new BoxLayout (tools, BoxLayout.X_AXIS));
    tools.setBackground(Color.yellow);
    tools.setOpaque(true);

    titleLabel = new JLabel("Heighway's Dragon");
    titleLabel.setForeground(Color.black);

    increase = new JButton (new ImageIcon("increase.gif"));
    increase.setPressedIcon(new
    ImageIcon("increasePressed.gif"));
    increase.setMargin(new Insets(0, 0, 0, 0));
    increase.addActionListener (this);

    decrease = new JButton (new ImageIcon("decrease.gif"));
    decrease.setPressedIcon(new
    ImageIcon("decreasePressed.gif"));
    decrease.setMargin(new Insets(0, 0, 0, 0));
```

```
    decrease.addActionListener(this);

    orderLabel = new JLabel("Order: 1");
    orderLabel.setForeground(Color.black);

    tools.add(titleLabel);
    tools.add(Box.createHorizontalStrut(20));
    tools.add(decrease);
    tools.add(increase);
    tools.add(Box.createHorizontalStrut(20));
    tools.add(orderLabel);

    drawing = new HeighwayDragonPanel(1);

    appletPanel = new JPanel();
    appletPanel.add(tools);
    appletPanel.add(drawing);

    getContentPane().add(appletPanel);

    setSize(APPLET_WIDTH, APPLET_HEIGHT);
  }

  // determines which button was pushed, and sets the new order
  // if it is in range
  public void actionPerformed(ActionEvent event){

    int order = drawing.getOrder();

    if(event.getSource() == increase)
      order++;
    else
      order--;

    if(order >= MIN && order <= MAX)
    {
      orderLabel.setText("Order: " + order);
      drawing.setOrder (order);
      repaint();
    }
  }
}
```

```
/***********************************
* Heighway's Dragon                *
* DragonPanel.java                 *
* Author: Lindsey Carr             *
* Date: Summer 2003                *
* Honor's Project                  *
***********************************/

import java.awt.*;
import javax.swing.JPanel;

public class HeighwayDragonPanel extends JPanel{

  // screen size
  private final int PANEL_WIDTH = 512;
  private final int PANEL_HEIGHT = 512;

  private int current; // current order

  // sets the initial fractal order to the value specified
  public HeighwayDragonPanel (int currentOrder){
    current = currentOrder;
    setBackground(Color.black);
    setPreferredSize(new Dimension(PANEL_WIDTH, PANEL_HEIGHT));
  }

  // draws the fractal recursively.  Base case is an order of 0
  // for which a simple straight line is drawn.
  // Otherwise, rotates and dilates the segments.
  public void drawFractal (int order, int x1, int y1, int x3,
                           int y3, Graphics page)
  {
    int x2, y2;

    if (order == 1) // base case
    {
      page.drawLine(x1, y1, x3, y3);
    }
    else
    {
      x2 = (x1 + x3 + y1 - y3) / 2;
      y2 = (-x1 + x3 + y1 + y3) / 2;

      drawFractal(order - 1, x1, y1, x2, y2, page);
      drawFractal(order - 1, x3, y3, x2, y2, page);
    }
  }
```

```
  // initial calls to the drawFractal method
  public void paintComponent(Graphics page){
    super.paintComponent(page);
    page.setColor(Color.green);

    drawFractal (current, 128, 128, 384, 128, page);
  }

  // sets the fractal order to the specified value
  public void setOrder (int order){
    current = order;
  }

  // returns the current order
  public int getOrder(){
    return current;
  }
}
```

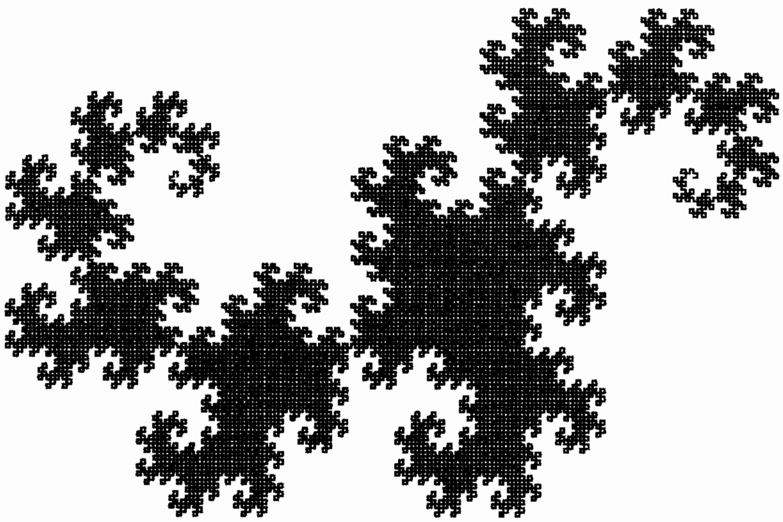


Figure A.1: Heighway's Dragon of order 15 generated using the Java code above

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